Announcements

1) HW 3 due Monday

2) Candidate Yulia Hristova
will speak sometime
tomorrow on her work
in numerical analysis
(2-3 PM 2070)

Theorem: (Bolzano-Weierstrass)

Any bounded sequence admits a convergent subsequence.

proof Let $\varepsilon > 0$. Recall from your homework that if $(a_n)_{n \in \mathbb{N}}$ is a bounded sequence and if

$$y_k = \sup\{a_n \mid n \geq k\}$$

you will show that $\exists$ $y \in \mathbb{R}$, $y_k \to y$ as $k \to \infty$. $\left(y = \limsup_{n\to\infty} a_n\right)$

The problem: we're done if $\exists$, for each k, an $n_k \in \mathbb{N}$ with $y_k = a_{n_k}$. Since y_k is defined as a supremum, we may not be able to find such an n_k!

The Solution: Define a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ inductively.

Since $y_k \longrightarrow y$ as $k \to \infty$,

$\exists$ a t_1 with

$$|y_{t_1} - y| < \frac{1}{2}$$

Then since $y_{t_1} = \sup\{a_n \mid n \geq t_1\}$,

$\exists$ an $n_1 \in \mathbb{N}$,

$$y_{t_1} - a_{n_1} < \frac{1}{2}$$

Then

$$|y - a_{n_1}|$$

$$= |y - y_{t_1} + y_{t_1} - a_{n_1}|$$

$$\leq |y - y_{t_1}| + |y_{t_1} - a_{n_1}|$$

(triangle inequality)

$$= |y - y_{t_1}| + (y_{t_1} - a_{n_1})$$

$$< \frac{1}{2} + \frac{1}{2} = 1$$

We know $\exists\ n_1$ with

$$|a_{n_1} - y| < 1.$$

Suppose we have inductively found $n_1 < n_2 < \cdots < n_k$

with

$$|a_{n_m} - y| < \frac{1}{m}$$

<u>Goal</u> Find $a_{n_{k+1}}$ with $\boxed{n_{k+1} > n_k}$

<u>and</u>

$$|a_{n_{k+1}} - y| < \frac{1}{k+1}.$$

Since $y_k \to y$ as $k \to \infty$,

$\exists$ S_k with

$$|y_m - y| < \frac{1}{2(k+1)}$$

$\forall$ $m \geq S_k$.

Choose $t_k = \max\{n_k, S_k\}$.

Since $y_{t_k} = \sup\{a_n \mid n \geq t_k\}$,

$\exists\, n_{k+1} > t_k$ so that

$$y_{t_k} - a_{n_{k+1}} < \frac{1}{2(k+1)}$$

Observe. $n_{k+1} > t_k = \max\{n_k, S_k\}$
$\geq n_k$

Then

$$|y - a_{n_{k+1}}|$$

$$= |y - y_{t_k} + y_{t_k} - a_{n_{k+1}}|$$

$$\leq |y - y_{t_k}| + |y_{t_k} - a_{n_{k+1}}|$$

$$= |y - y_{t_k}| + (y_{t_k} - a_{n_{k+1}})$$

$$< \frac{1}{2(k+1)} + \frac{1}{2(k+1)}$$

$$= \frac{1}{k+1}.$$

So we have found a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ so that

$$|a_{n_k} - y| < \frac{1}{k}$$

$\forall\ k \in \mathbb{N}$.

If $\varepsilon > 0$, $\exists\ K \in \mathbb{N}$, $\frac{1}{\varepsilon} < K$. Hence, $\frac{1}{K} < \varepsilon$, so $\forall\ k \geq K$,

$$|a_{n_k} - y| < \frac{1}{k} < \frac{1}{K} < \varepsilon. \quad \square$$

Example 1: Note that

for $a_n = \frac{1}{n}$,

$y_k = \sup\{a_n \mid n \geq k\} = \frac{1}{k} = a_k$,

so y_k is a term of this sequence $\forall k \in \mathbb{N}$. But with

$a_n = 1 - \frac{1}{n}$, $y_k = 1 \ \forall k \in \mathbb{N}$,

and no term of the sequence is ever equal to one

Therefore, you need the argument in the previous theorem!

(Can't choose $a_{n_k} = y_k$)

Note: The same proof works for liminf in place of limsup.

So we obtain as a consequence of the proof the following

Corollary Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence.

Let $y = \limsup_{n\to\infty} a_n$, $z = \liminf_{n\to\infty} a_n$

Then $\exists$ subsequences of $(a_n)_{n\in\mathbb{N}}$ converging to both y and z.

More Properties of Sequences

2) If $\lim_{n\to\infty} a_n = L$, and $c \in \mathbb{R}$,

$$\lim_{n\to\infty} (c a_n) = cL.$$

proof: Let $\varepsilon > 0$. We want $|ca_n - cL| < \varepsilon$

We can factor out

$$|ca_n - cL| = |c(a_n - L)|$$
$$= |c|\,|a_n - L|$$

If $c=0$, then $ca_n = cL = 0$, and the convergence is clear.

If $c \neq 0$, then

$$|c|\,|a_n - L| < \varepsilon \text{ when}$$

$$|a_n - L| < \frac{\varepsilon}{|c|}.$$

Since $a_n \to L$ as $n \to \infty$, $\exists$ $N \in \mathbb{N}$,

$$|a_n - L| < \frac{\varepsilon}{|c|}, \text{ hence}$$

$$|ca_n - cL| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \quad \square.$$

3) If $\lim_{n\to\infty} a_n = L$ and

$\lim_{n\to\infty} b_n = M$, then

$$\lim_{n\to\infty} a_n b_n = LM.$$

proof: Let $\varepsilon > 0$ we want

$$|a_n b_n - LM| < \varepsilon$$

Use triangle inequality

$$|a_n b_n - L b_n + L b_n - LM|$$

$$= |a_n b_n - LM|$$

$$|a_n b_n - L b_n + L b_n - LM|$$

$$\leq |a_n b_n - L b_n| + |L b_n - LM|$$

$$= |b_n|\,|a_n - L| + |L|\,|\underbrace{b_n - M}|$$

what about this term?

can make this $< \frac{\varepsilon}{L}$

Next time: more sequences, section 2.6.